Derivatives of Inverse Functions

Theorem: Let f be a function whose domain is an interval I. If f has an inverse function, then the following are true:

- If f is continuous on its domain, then f^{-1} is continuous on its domain
- If f is differentiable at c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$

Derivative of Inverse Function: Let f be a function that is differentiable on I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. And $\boldsymbol{g'(x) = \frac{1}{f'(g(x))}, f'(g(x)) \neq 0}$.

*** Graphs of Inverse Functions have Reciprocal Slopes :

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Example: If $f(x) = x^3 + 2x - 10$, find $(f^{-1})'(x)$.

Method 1:

(a) Establish that the function is differentiable and has an inverse, then we know that the derivative of the inverse exists.

(b) Let $y = f(x)$

(c) Interchange x and y to obtain the inverse function

(d) Differentiate with respect to y: $\frac{dx}{dy} =$

(e) Apply formula: $\frac{dy}{dx} =$

Method 2: (Implicit Differentiation)

(a) Let $y = f(x)$

(b) Interchange x and y to obtain the inverse function

(c) Differentiate each term *implicitly* with respect to x

(d) Solve for $\frac{dy}{dx}$.

Example: If $f(x) = 2x^5 + x^3 + 1$. Find (a) $f(1)$ (b) $f'(1)$ (c) $(f^{-1})(4)$ (d) $(f^{-1})'(4)$

$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$	$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$	$\frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$
$\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$	$\frac{d}{dx}[\text{arccsc}\, u] = \frac{-u'}{\lvert u\rvert\sqrt{u^2-1}}$

Example: Differentiate $y = 5\sin^{-1}(3x)$

Example: Differentiate $y = \sec^{-1}(3x^2)$

Example: Differentiate $y = \arcsin x + x\sqrt{1-x^2}$